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ON ISOLATED GAPS OF NON-PERFECT ARF NUMERICAL SEMIGROUPS

Meral SÜER [1*], Ravan ERTUĞRUL [2]

[1]*Batman University, Department of Mathematics, Faculty of Sciences and Letters, Batman, Türkiye*
[2]*Batman University, Department of Mathematics, Institute of Graduate Studies, Batman, Türkiye*
**** Corresponding Author:*** *meral.suer@batman.edu.tr*

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ABSTRACT

This study examines the structure of non-perfect Arf numerical semigroups and explores the characteristics of their isolated gaps. The analysis focuses on the structural relationship between the elements of Arf numerical semigroups and their gap sets. For non-perfect Arf numerical semigroups, the position and behavior of isolated gaps, together with their connection to the Frobenius number, are characterized. It is shown that the union of an Arf numerical semigroup with its set of isolated gaps forms a perfect Arf numerical semigroup. Moreover, when certain subsets of isolated gaps are considered, the resulting semigroups preserve their non-perfect nature. These findings shed new light on the relationship between perfect and non-perfect Arf numerical semigroups and clarify the role of isolated gaps in the transition between these two classes.

1. INTRODUCTION

Let $\mathbb{N} = \{0,1,2,3, \dots\}$ and $S \subseteq \mathbb{N}$. A set S is called a numerical semigroup if it contains 0, is closed under addition, and its complement $\mathbb{N} \setminus S$ is finite. When S is a numerical semigroup and $A \subseteq S$, then A is a generating set of S if

$$S = \langle A \rangle = \left\{ \sum_{i=1}^{k} s_i a_i : s_i \in A, \qquad a_i \in \mathbb{N} \right\}. \tag{1}$$

Moreover, if $S =< s_1, s_2, \dots, s_n >$ and no proper subset of $A = \{s_1, s_2, \dots, s_n\}$ generates S, then A is called the minimal generating system of S. The smallest element of a minimal generating system is referred to as the multiplicity of S and is denoted by $m(S)$.

For a numerical semigroup S, the complement $\mathbb{N} \setminus S$ is called the set of gaps of S and is denoted by $G(S)$. The cardinality of $G(S)$ is the genus of S, written $g(S)$, while the largest element of $G(S)$ is called the Frobenius number and is denoted by $F(S)$. In particular, the Frobenius number of $\mathbb{N}$ is -1.

A numerical semigroup S is considered a proper numerical semigroup if it is not equal to $\mathbb{N}$. The numerical semigroups we will consider throughout this study are proper numerical semigroups.

All standard definitions and basic properties of numerical semigroups used in this section can be found in [1].

The set of isolated gaps of S, denoted by $I(S)$, is defined as the set of $g \in G(S)$ such that $g-1, g+1 \in S$. A numerical semigroup without isolated gaps is referred to as perfect [2-3].

A numerical semigroup S is an Arf numerical semigroup if for all $x, y, z \in S$ with $x \geq y \geq z$, it holds that $x + y - z \in S$ or $2x - y \in S$ [4-5].

Numerical semigroups have been extensively studied due to their rich algebraic structure and applications in areas such as algebraic geometry, coding theory, and combinatorics [6].

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Recent advances in the theory of numerical semigroups have emphasized the structural and arithmetic characterization of specific subclasses defined by perfection and gap distribution. Moreno-Frais and Rosales [2, 7] developed a systematic study of perfect numerical semigroups, establishing necessary and sufficient conditions for their existence and describing their relationships with fundamental invariants such as the Frobenius number and the Apéry set. They also examined perfect numerical semigroups with maximal embedding dimension, including notable classes such as Arf and saturated numerical semigroups. Building upon these developments, Smith [8] developed methods for finding all isolated gaps of a numerical semigroup of embedding dimension two and showed how to produce perfect numerical semigroups in three dimensions. Furthermore, Smith [3] explored quotients of perfect numerical semigroups, highlighting how quotient operations preserve or modify the perfection property and affect key numerical invariants. İlhan [9] focuses on the classification of perfect Arf numerical semigroups and some of their properties. Collectively, these contributions deepen the understanding of how algebraic properties and combinatorial structures interplay within numerical semigroups, inspiring new directions for future research.

While the study of perfect Arf numerical semigroups has garnered significant attention, the exploration of non-perfect Arf numerical semigroups, particularly concerning their isolated gaps, remains underdeveloped. Previous works have characterized the isolated gaps in numerical semigroups of embedding dimension two and examined the properties of perfect numerical semigroups with fixed Frobenius numbers [8-10].

However, the interplay between the Arf property and the presence of isolated gaps in non-perfect semigroups has not been extensively investigated.

This study aims to bridge this gap by providing a comprehensive analysis of isolated gaps in non-perfect Arf numerical semigroups. By examining the conditions under which isolated gaps occur and their implications for the structure of the semigroup, this research contributes to a deeper understanding of the combinatorial and algebraic properties of Arf numerical semigroups.

In summary, while significant progress has been made in understanding perfect Arf numerical semigroups, the study of isolated gaps in non-perfect Arf numerical semigroups is a relatively unexplored area. This research seeks to fill this gap, offering new insights into the intricate structure of these semigroups and paving the way for further investigations in this domain.

2. MAIN RESULTS

Proposition 2.1. [11, Lemma 1] Let S be an Arf numerical semigroup. If $x, x+1 \in S$, then every integer $g \in \mathbb{Z}$ with $g > x+1$ also belongs to S; that is, $g \notin G(S)$.

Proof. Suppose S is an Arf numerical semigroup and $x, x+1 \in S$. By the Arf property, choosing $a = b = x+1$ and $c = x$ we obtain

$$2(x+1) - x = x + 2 \in S. \tag{2}$$

Iterating this argument shows inductively that all subsequent integers $x+3, x+4,$ are contained in S. Therefore every integer $g \in \mathbb{Z}$ with $g > x+1$ necessarily lies in S; equivalently, no gap of S exceeds $x+1$.

Proposition 2.2. Let S be a non-perfect Arf numerical semigroup, and let $I(S)$ denote the set of isolated gaps of S. Denote by h the smallest element of $I(S)$. Then, for every integer x with $x > h$, one has $x \in S$ or $x \in I(S)$. Moreover, the Frobenius number $F(S)$ belongs to $I(S)$.

Proof. Since $h \in I(S)$, it follows by definition that both $h-1$ and $h+1$ belong to S. As S is Arf, we may apply the Arf property with $x = y = h+1$ and $z = h-1$, which yields

$$2(h+1) - (h-1) = h + 3 \in S. \tag{3}$$

Consequently, $h+2$ is either an element of S or a gap of S. In the latter case, since $h+1, h+3 \in S$, the integer $h+2$ must be an isolated gap; that is, $h+2 \in I(S)$.

We now distinguish two cases.

Case 1: $h+2 \in S$.

In this situation, since $h+1 \in S$, Proposition 2.1 ensures that every integer $x > h$ must lie in S. Consequently, h is the largest gap of S, so $F(S) = h$.

Thus, $F(S) \in I(S)$.

Case 2: $h+2 \in I(S)$.

Because $h+1, h+3 \in S$, another application of the Arf property shows that $h+5 \in S$. It follows that $h+4$ must lie either in S or in $I(S)$. Repeating this reasoning, we deduce that all integers greater than $h+4$ either belong to S or eventually force the inclusion of $h+7 \in S$.

Since S has only finitely many gaps, this process must terminate. Therefore, the Frobenius number $F(S)$ necessarily belongs to $I(S)$.

Corollary 2.3. (Structure of isolated gaps of Arf numerical semigroups). Let S be a non-perfect Arf numerical semigroup and denote by $I(S)$ the set of isolated gaps of S. Then the isolated gaps form an arithmetic progression of Case 2; more precisely there exist integers h and $k \geq 1$ such that

$$I(S) = \{\, h,\ h+2,\ \ldots,\ h+2(k-1) \,\}, \tag{4}$$

and $h+2(k-1) = F(S)$ the Frobenius number of S.

Proof. This is an immediate consequence of Proposition 2.2. Indeed Proposition 2.2 establishes that, with h taken as the minimal element of $I(S)$, every integer $x > h$ is either in S or in $I(S)$, and the iterative Arf-argument used in that proposition forces any two distinct isolated gaps to differ by an even integer. Since $G(S)$ is finite, the isolated gaps above h must therefore form a finite arithmetic progression with common difference 2, terminating at the largest gap $F(S)$.□

Remark 2.4. It is essential to state that the union of the semigroup S and its set of isolated gaps $I(S)$, denoted by $S \cup I(S)$, itself forms a semigroup. This fact is established in the literature [8].

Theorem 2.5. Let S be a non-perfect Arf numerical semigroup and let $I(S)$ denote the set of isolated gaps of S. Then the union $S \cup I(S)$ is a perfect Arf numerical semigroup.

Proof. By Corollary 2.3, the isolated gaps of S have the form

$$I(S) = \{\, h,\ h+2,\ h+4,\ \ldots,\ h+2(k-1) = F(S) \,\} \tag{5}$$

for some integer h and $k \geq 1$. We will show (i) $S \cup I(S)$ satisfies the Arf property, and (ii) $S \cup I(S)$ contains no isolated gaps; together (i) and (ii) imply that $S \cup I(S)$ is a perfect Arf numerical semigroup.

(i) **Arf property.** Let $x, y \in S \cup I(S)$ with $x \geq y$. We must prove $2x - y \in S \cup I(S)$. There are three mutually exhaustive cases.

- **Case 1:** $x, y \in S$.

Since S is Arf, $2x - y \in S \subseteq S \cup I(S)$.

- **Case 2:** $x, y \in I(S)$.

By the definition of isolated gaps, $x \pm 1, y \pm 1 \in S$. Using the Arf property in S on suitable triples of these neighbours we deduce that the two integers adjacent to $2x - y$, namely $2x - y - 1$ and $2x - y + 1$, lie in S. Hence either $2x - y \in S$ or, if not, the adjacency condition forces $2x - y$ to be an isolated gap, i.e. $2x - y \in I(S)$. In either event $2x - y \in S \cup I(S)$.

- **Case 3:** $x \in S$ and $y \in I(S)$ (the symmetric case $x \in I(S), y \in S$ is identical up to swapping).

As $y \in I(S)$ we have $y \pm 1 \in S$. Applying the Arf property in S to the triples

$x' = x, y' = x, z' = y - 1$ (i.e. $x' \geq y' \geq z'$) and $x'' = x, y'' = x, z'' = y - 1$ (i. e. $x'' \geq y'' \geq z''$) yields

$$2x - y + 1 \in S, 2x - y - 1 \in S. \tag{6}$$

Thus again the two neighbours of $2x - y$ are in S, so $2x - y \in S$ or $2x - y \in I(S)$, whence $2x - y \in S \cup I(S)$.

All cases produce $2x - y \in S$ establishing that $S \cup I(S)$ is an Arf numerical semigroup.

(ii) **Absence of isolated gaps.** By Proposition 2.2, if h is the smallest isolated gap of S, then contains all nonnegative integers $x \leq (h - 1)$, elements of which are less than $h - 1$ are exactly the elements of S which are less than $h - 1$. Gaps of are exactly gaps of S which are less than h, none of which is isolated. Equivalently, $S \cup I(S)$ contains no isolated gaps and hence is perfect.

Combining (i) and (ii) we conclude that $S \cup I(S)$ is a perfect Arf numerical semigroup of Frobenius number less than h.

Theorem 2.6. Let S be a non-perfect Arf numerical semigroup and let $I(S)$ denote the set of isolated gaps of S. Write $h = minI(S)$ and $k = \#I(S)$ (Here #A denotes the cardinality of a set A). For each $i = 0,1, \dots, k - 1$, define

$$I^{(i)}(S) = \{h + 2i, h + 2(i + 1) \dots, h + 2(k - 1) = F(S)\} \tag{7}$$

Then, for every $i > 0$, the set $S \cup I^{(i)}(S)$ is a non-perfect Arf numerical semigroup.

Proof. By the Corollary preceding Corollary 2.3, the set of isolated gaps of S is of the form

$$I(S) = \{\, h,\ h + 2,\ \dots,\ h + 2(k - 1) = F(S)\,\}, \tag{8}$$

a finite arithmetic progression of even step size. By definition $I^{(i)}(S) \subset I(S)$ and consists only of the last part of this sequence. From Theorem 2.5 we know that $S \cup I(S)$ is a perfect Arf numerical semigroup, since adjoining all isolated gaps eliminates them entirely.

However, when $i > 0$, the set $I^{(i)}(S)$ is a proper subset of $I(S)$. Hence

$$S \cup I^{(i)}(S) \subsetneq S \cup I(S), \tag{9}$$

and in particular, the elements $\{h, h + 2, \dots, h + 2(i - 1)\}$ have not been added. Consequently, the union $S \cup I^{(i)}(S)$ still contains isolated gaps and therefore remains non-perfect.

Furthermore, since $h + 2i - 1$ and all subsequent integers belong to $S \cup I^{(i)}(S)$, this union is a numerical semigroup. Each element $x \in I^{(i)}(S)$ satisfies $x - 1, x + 1 \in S$ by construction. To show that $S \cup I^{(i)}(S)$ is still an Arf numerical semigroup, we observe the following:

- For any $x, y \in S$, we have $2x - y \in S$, as S is Arf.
- For $x \in S$, $y \in I^{(i)}(S)$, the definition of isolated gaps ensures $y \pm 1 \in S$. Using the Arf condition on these neighbours allows us to deduce $2x - y \in \mathrm{S} \cup I^{(i)}(S)$.
- Similarly, for $x, y \in I^{(i)}(S)$ with $x > y$, we can derive $2x - y \in \mathrm{S} \cup I^{(i)}(S)$ by applying the Arf condition to neighbouring elements in S.

Each of these follows by the same arguments as in the proof of Theorem 2.5.

Thus, $S \cup I^{(i)}(S)$ is an Arf numerical semigroup but, since it still possesses isolated gaps, it is not perfect.

This completes the proof.

3. CONCLUSION AND SUGGESTIONS

In this work, we examined the structural properties of non-perfect Arf numerical semigroups through the behavior of their isolated gaps.

We proved that if S is a non-perfect Arf numerical semigroup and $I(S)$ denotes the set of its isolated gaps, then the union $S \cup I(S)$ forms a perfect Arf numerical semigroup.

This result provides an explicit and constructive method to obtain perfect Arf numerical semigroups directly from non-perfect ones, highlighting the fundamental role of isolated gaps in achieving perfection.

Furthermore, by defining the family

$$I^{(i)}(S) = \{h + 2i, h + 2(i + 1)\ldots, h + 2(k - 1) = F(S)\}, \tag{10}$$

we have shown that, for each $i > 0$, the set $S \cup I^{(i)}(S)$ remains an Arf numerical semigroup, but it is non-perfect due to the persistence of some isolated gaps.

This hierarchical construction establishes a transition framework between non-perfect and perfect Arf numerical semigroups, deepening the understanding of how isolated gaps influence the structural completeness of S. The results collectively demonstrate that isolated gaps are not merely random irregularities within a semigroup but algebraically controllable features. Their systematic inclusion restores closure under the Arf condition and reveals the internal stability mechanisms of numerical semigroups.

Future research may pursue several directions inspired by these results:

- Develop numerical or combinatorial invariants that quantify how far a given Arf numerical semigroup is from being perfect, possibly through the number or distribution of its isolated gaps.
- Investigate the poset or lattice structure formed by the chain

$$S \subset S \cup I^{(k-1)}(S) \subset \cdots \subset S \cup I^{(i)}(S) \tag{11}$$

and explore how Arf invariants behave monotonically along this hierarchy.

- Extend the framework to higher-dimensional analogues-such as Arf semirings or numerical monoids-and study how the concept of isolation translates to these contexts.

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Contributions of the Authors

Meral SÜER: Conceptualization, formal analysis, investigation, methodology, resources, writing – original draft, writing – review and editing.

Ravan ERTUĞRUL: Methodology, resources, data curation, investigation, writing – review and editing.

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